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Note

On Chains and Sperner k-Families in Ranked Posets, II

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A ranked poset P satisfies condition S if for all k the set of elements of the k largest ranks in P is a Sperner k-family. It satisfies condition T if for all k there exist disjoint chains in P which each meet the k largest ranks and which cover the kth largest rank. Griggs employed the theory of saturated partitions to prove that if P satisfies S it also satisfies T, and that the converse is true for posets with unimodal Whitney numbers. Here we present new proofs of these facts which do not require the existence of saturated partitions. The first result is proven with a simple network flow argument and the second is proven directly.

In [3] Griggs introduced several conditions on ranked posets P and studied their relationships to each other. The results of most interest were that condition S, which is a generalization of the Sperner property, and condition T, which conerns the existence of certain sets of chains of P, are strongly related. Specifically, if S holds in P, then T holds in P, and the converse, while false in general, is true if P is restricted to have unimodal Whitney numbers. The proofs of these results depended on deep results of Greene and Kleitman [2] concerning the existence of saturated partitions of posets. In this paper we present new proofs of these results relating S and T without relying on saturated partition theory. That T holds in P if S holds is shown by a simple network flow argument. That S hold in a poset P with

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unimodal Whitney numbers, if T holds, is proven directly. We first establish the notation and review some definitions. See [3] or [2] for more details.

Throughout the paper, P shall denote a ranked (or graded) poset of rank n. The set of elements in P of rank i, $0 \leqslant i \leqslant n$, is denoted P_i. The number of elements of rank i, $|P_i|$, is called the ith Whitney number and is denoted N_i. $(N_0, N_1, \ldots, N_n)$ is the sequence of Whitney numbers of P. If for some i $N_0 \leqslant N_1 \leqslant \cdots \leqslant N_i \geqslant N_{i+1} \geqslant \cdots \geqslant N_n$, then the sequence is said to be unimodal. $(M_0, M_1, \ldots, M_n)$ is the sequence of Whitney numbers listed in decreasing order. A subset A of P is a k-family if no $k+1$ elements lie on a chain in P. Equivalently, a k-family is the union of k antichains. A Sperner k-family is a k-family of maximum size.

The k largest ranks in P form a k-family of size $M_0 + \cdots + M_{k-1}$. If it is a Sperner k-family, then P satisfies *condition* S_k. S_1 is just the Sperner property. If P satisfies S_k for all k, it is said to satisfy *condition* S.

P satisfies *condition* T_k, $0 \leqslant k \leqslant n$, if there exists a set of M_k disjoint chains in P which each intersect each rank of size at least M_k. P satisfies *condition* T if T_k holds in P for all k.

THEOREM 1. *If a ranked poset P satisfies condition S, then it satisfies condition T.*

Proof. Suppose P satisfies condition S. It clearly suffices to show that T_k holds for those k such that $M_k > M_{k+1}$ and for $k = n$. Let k assume one of these values. T_k follows from S_k by the following network flow argument. (Refer to [1] for details on network flows.)

Let P' consist of the set of elements of the $k+1$ largest ranks in P, ordered as in P. P' is ranked, has rank k, and satisfies S_k. To show P satisfies T_k, we need to show that there exist M_k disjoint chains in P' which meet every rank. (In the original proof [3], this follows immediately from the existence of a k-saturated partition of P'.)

Let $\mathcal{N}$ be this network:

	Nodes
A	source
B	sink
U_x	(for all $x \in P'$)
V_x	(for all $x \in P'$)

Arcs		*Capacities*
(A, U_x)	(for all $x \in P'$ of rank 0)	$+\infty$
(V_x, B)	(for all $x \in P'$ of rank k)	$+\infty$
(U_x, V_x)	(for all $x \in P'$)	1
(V_x, U_y)	(for all $x, y \in P'$ such that y covers x)	$+\infty$

"y covers x" means $y > x$ in P' and no element is between x and y. $\mathcal{N}$ has capacity at most M_k because the arcs (U_x, V_x) corresponding to the elements x of a rank of size M_k in P' form a cut-set with capacity M_k. Indeed, the capacity of $\mathcal{N}$ is precisely M_k by condition S_k. For let C be a minimum capacity cut-set of $\mathcal{N}$. It must consist of arcs of the type (U_x, V_x). Let $\bar{C}$ denote the set of elements of P' corresponding to these arcs in C. Deleting C from $\mathcal{N}$ cuts all flows through $\mathcal{N}$. In P' this means that no chain in $P' - \bar{C}$ meets all $k+1$ ranks. Hence $P' - \bar{C}$ is a k-family. It follows from condition S_k that $|\bar{C}| \geqslant M_k$, so that C has capacity at least M_k.

By the Max Flow-Min Cut theorem, $\mathcal{N}$ has a flow of size M_k. $\mathcal{N}$ has integral capacities (replacing $+\infty$ by a sufficiently large integer, such as $|P|+1$), so by the Integrity Theorem there must be an integer-valued maximum flow. By the design of $\mathcal{N}$, this integer flow of size M_k consists of M_k single unit flows which correspond naturally to M_k disjoint chains in P' which meet every rank. ∎

THEOREM 2. *If P is a ranked poset with a unimodal sequence of Whitney numbers and if P satisfies condition T, then P satisfies condition S.*

Proof. Suppose P satisfies the conditions of the theorem. By the unimodularity, there exists an ordering $a(i)$ of the ranks such that

$$N_{a(0)} \geqslant N_{a(1)} \geqslant \cdots \geqslant N_{a(n)},$$

and such that for all i, $a(i)$ is adjacent to one of $a(0), a(1),..., a(i-1)$. We must show for each k that S_k holds.

Let A be a Sperner k-family in P. It suffices to show that

$$|A| = N_{a(0)} + N_{a(1)} + \cdots + N_{a(k-1)}.$$

If $A = P_{a(0)} \cup P_{a(1)} \cup \cdots \cup P_{a(k-1)}$, this is immediate. Therefore suppose A does not equal this union. Let $I = \min\{a(0), a(1),..., a(k-1)\}$. It must be true that $A \cap P_i \neq \varnothing$ for some $i < I$ or $i > I+k-1$. Suppose first that $A \cap P_i \neq \varnothing$ for some $i < I$. Assume i is minimum such that $A \cap P_i \neq \varnothing$. We show below how to push elements of $A \cap P_i$ into higher ranks, preserving the size of A and the fact that A is a k-family. $A \cap P_i$ never gets pushed above rank $I+k-1$. This procedure can then be applied to $A \cap P_{i+1},...,$ $A \cap P_{I-1}$, to push A up into ranks I and above. The dual procedure works in the same way to pull A down from ranks above ${}_{I+k-1}$. Note that the dual of P need not be ranked at all, but the dual of the procedure below works just the same. At the end A is confined to the k largest ranks and S_k must hold.

By the unimodularity of the N_j, it must be true that

$$N_i \leqslant N_{i+1}, N_{i+2},..., N_{I+k-1}.$$

By condition T, there exists a set of N_i disjoint chains meeting each of $P_i, P_{i+1},..., P_{I+k-1}$. For each $x \in A \cap P_i$ let C_x be the chain in this set which contains x. Replace each $x \in A \cap P_i$ by the lowest element x' on C_x which lies above x but does not already belong to A. Such an element x' exists and has rank at most $I+K-1$ since $|A \cap C_k| \leqslant k$. Let A' be the set obtained from A by pushing up each element $x \in A \cap P_i$.

Clearly $|A'| = |A|$, so it remains to prove that A' is a k-family. Let C be a chain in P. We need to show that $|A' \cap C| \leqslant k$. This follows from $|A \cap C| \leqslant k$ if $(A' - A) \cap C = \varnothing$. If $(A' - A) \cap C \neq \varnothing$, let y be maximum in $(A' - A) \cap C$. Let l be the rank of y in P. y is x' for some $x \in A \cap P_i$. By construction, the chain C_x contains y and intersects A at elements of ranks $i, i+1,..., l-1$. So $l-i$ elements of A lie on C_x below y. Hence at most $k-l+i$ elements of A lie on C above y because the chain formed by combining the part of C above y with the part of C_x below y meets the k-family A at most k times. By the maximality of y, A' meets C above y at most $k-l+i$ times. Since A' lies entirely above rank i, C meets A' below y at most $l-i-1$ times. Adding up, we conclude that $|A' \cap C| \leqslant k$. ∎

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